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Weight Function Approach to q-Difference Equations for the q-Hypergeometric Polynomials

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In this paper we define a new algebra generated by the difference operators D_q and $D_{q^{-1}}$ with two analytic functions $\alpha(x)$ and $\beta(x)$. Also, we define an operator $M = J_1 J_2 - J_3 J_4$ s.t. all q-hypergeometric orthogonal polynomials $Y_n(x)$, $x \neq \cos(\theta)$, are eigenfunctions of the operator M with eigenvalues $\lambda_q [n]_q$. The choice of $\alpha(x)$ and $\beta(x)$ depend on the weight function of $Y_n(x)$.

KEY WORDS: weight function; q-difference equations; q-hypergeometric polynomials.

1. INTRODUCTION

It is known that there is a relation between Lie groups and certain special functions (Miller, 1968). Special functions appear as basis vectors and matrix elements corresponding to a local multiplier representations of Lie groups (Vilekin, 1968). For example Bessel functions appear in two distinct ways: as matrix elements of a local irreducible representation of $G(0, 0)$ and as basis functions for an irreducible representation of the four-dimensional Lie algebra $g(0, 0)$. So Lie theory gives a natural setting for an algebraic interpretation of the special functions.

The q-special functions are extensions to a base q of the standard special functions. A connection between q-special functions and quantum algebra has been established (Gonzalez and Ibort, 1992; Koelink, 1996; Koornwinder, 1992, 1994). In this case one considers matrix elements of operators built with q-exponentials of operators; these elements turn out to be expressible in terms of q-hypergeometric series. Also q-special functions appear as the basis of irreducible representations of quantum algebras.

In this paper, in analogy with the Lie theoretic treatment of standard functions, we suggest a new approach to studying the relation between quantum groups

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and q-special functions we consider a new algebra generated by the operators J_1, J_2, J_3, J_4, we call it D_q-algebra. On the D_q-algebra we will define a new operator $M = J_1 J_2 - J_3 J_4$ such that all q-hypergeometric orthogonal polynomials $Y_n(x)$, $x \neq \cos(\theta)$, are eigenfunctions of the operator M with eigenvalues $\lambda_q[n]_q$, that is

$$MY_n = \lambda_q[n]_q Y_n,$$

where the quantum number $[n]_q$ is defined by

$$[n]_q = \frac{1-q^n}{1-q}.$$

We consider the operator M as a q-revised form of the Casimir operator in the algebra D_q. Also the choice of J_1 and J_3 will depend on the weight function of Y_n but J_2 and J_4 are the difference operators D_q and $D_{q^{-1}}$ (respectively).

2. BASIC CONCEPTS

The q-shifted factorials $(a;q)_k$ (the q-extension of the Pochhammer-symbol $(a)_k$) is defined by

$$(a;q)_k = \prod_{i=0}^{k-1}(1-aq^i),$$

with the following properties

$$(qx;q)_n = \frac{(1-q^n x)}{(1-x)}(x;q)_n,$$

$$(q^{-1}x;q)_n = \frac{(1-q^{-1}x)}{(1-q^{n-1}x)}(x;q)_n,$$

$$(q;q)_{n-r} = (-1)^r q^{\binom{r}{2}-nr}\frac{(q;q)_n}{(q^{-n};q)_r}.$$

Also

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

The basic hypergeometric series is defined by (Koekoek and Swarttouw, 1994)

$${}_r\varphi_s\left(\begin{matrix} a_1,\ldots,a_r \\ \\ b_1,\ldots,b_s \end{matrix} |q;x\right) = \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\frac{k}{2}(k-1)}\right)^{1+s-r}\frac{x^k}{(q;q)_k},$$

where

$$(a_1, \ldots, a_r; q)_k = \prod_{i=1}^{r} (a_i; q)_k.$$

The basic hypergeometric series ${}_r\varphi_s$ is a polynomial in x if one of a_i equals q^{-n}, n is a nonnegative integer. Otherwise the radius of convergence of ${}_r\varphi_s$ is

$$\rho = \begin{cases} \infty, & \text{if } r < s+1 \\ 1, & \text{if } r = s+1 \\ 0, & \text{if } r > s+1 \end{cases}$$

The classical exponential function e^x can be expressed in terms of the hypergeometric functions $e^x = {}_0F_0(\overline{-}; x)$ this function has two different natural q-extension denoted by $e_q(x)$ and $E_q(x)$ defined by

$$e_q(x) = {}_1\varphi_0 \left(\begin{matrix} 0 \\ - \end{matrix} \middle| q; x \right) = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty},$$

and

$$E_q(x) = {}_0\varphi_0 \left(\begin{matrix} - \\ - \end{matrix} \middle| q; x \right) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k} = (-x;q)_\infty,$$

where $x \in C$, $|x| < 1$, and $0 < q < 1$. Also $e_q(x)$ and $E_q(x)$ can be considered as formal power series in the formal variable x. They have the following properties:

$$\begin{aligned}
e_q(x)E_q(-x) &= 1, \\
e_q(qx) &= (1-z)e_q(x), \\
E_q(x) &= (1+x)E_q(qx), \\
e_q(x) &= (1-q^{-1}x)e_q(q^{-1}x), \\
E_q(q^{-1}x) &= (1+q^{-1}x)E_q(x), \\
e_q(q^n x)E_q(-x) &= (x;q)_n, \\
e_q(x)E_q(-q^n x) &= 1/(x;q)_n, \\
\lim_{q\to 1} e_q((1-q)x) &= \lim_{q\to 1} E_q((1-q)x) = e^x.
\end{aligned}$$

3. THE D_q-ALGEBRA

Definition 3.1. The q-difference operators D_q and D_{q-1} are defined by

$$D_{q^{\pm 1}} f(x) = \begin{cases} \dfrac{f(x) - f(q^{\pm 1}x)}{x(1-q^{\pm 1})}, & x \neq 0 \\ \dfrac{df(0)}{dx}, & x = 0 \end{cases}$$

where

$$\lim_{q \to 1} D_{q^{\pm 1}} f(x) = \frac{df(x)}{dx}.$$

The two q-exponentials $e_q(x)$ and $E_q(x)$ are eigenfunctions of the q-difference operators D_q and $D_{q^{-1}}$ (respectively) where

$$D_q e_q(x) = \frac{1}{1-q} e_q(x),$$

$$D_{q^{-1}} E_q(x) = \frac{1}{q-1} E_q(x).$$

Lemma. 3.2. *The two operators D_q and $D_{q^{-1}}$ are a quantum representation of the quantum plane algebra generated by x and y with the commutation relation $xy = qyx$ by putting*

$$x \to D_{q^{-1}} \quad \textit{and} \quad y \to D_q.$$

Proof:

$$\begin{aligned} D_q D_{q^{-1}} f(x) &= D_q \left(\frac{f(x) - f(q^{-1}x)}{(1-q^{-1})x} \right) \\ &= \frac{1}{(1-q)x} \left(\frac{f(x) - f(q^{-1}x)}{(1-q^{-1})x} - \frac{f(qx) - f(x)}{(1-q^{-1})qx} \right) \\ &= \frac{1}{(1-q)x} \left(D_{q^{-1}} f(x) - \frac{f(qx) - f(x)}{(q-1)x} \right) \\ &= \frac{1}{(1-q)x} (D_{q^{-1}} f(x) - D_q f(x)). \end{aligned}$$

Similarly

$$D_{q^{-1}} D_q f(x) = \frac{q}{(1-q)x} (D_{q^{-1}} f(x) - D_q f(x)).$$

Then we have

$$D_{q^{-1}} D_q = q D_q D_{q^{-1}}. \qquad \square$$

Definition 3.3. The D_q-algebra is a nonassociative algebra generated by the operators J_1, J_2, J_3, and J_4:

$$J_1 = \alpha(x), \qquad J_2 = D_{q^{-1}},$$
$$J_3 = \beta(x), \qquad J_4 = D_q,$$

with the commutation relations

$$\begin{aligned}
J_1 J_2 &= (1 - q^{-1})x(J_2 J_1)J_2,\\
J_1 J_4 &= (1 - q)x(J_4 J_1)J_4,\\
J_1 J_3 &= J_3 J_1,\\
J_3 J_2 &= (1 - q^{-1})x(J_2 J_3)J_2,\\
J_2 J_4 &= q J_4 J_2,\\
J_3 J_4 &= (1 - q)x(J_4 J_3)J_4,
\end{aligned}$$

where the functions $\alpha(x)$ and $\beta(x)$ are analytic functions in the variable x.

Definition 3.4. On the D_q-algebra define the operator M by

$$M = J_1 J_2 - J_3 J_4$$

such that all q-hypergeometric orthogonal polynomials $Y_n(x)$, $x \neq \cos(\theta)$, are eigenfunctions of the operator M with eigenvalues $\lambda_q[n]_q$.

And the operator M satisfies the following relations for any analytic function $\psi(x)$:

$$\begin{aligned}
M(J_1\psi(x)) &= (MJ_1)\psi(x),\\
M(J_3\psi(x)) &= (MJ_3)\psi(x),\\
J_2 M\psi(x) &= [(J_2 J_1)J_2 - (J_2 J_3)J_4]\psi(x),\\
J_4 M\psi(x) &= [(J_4 J_1)J_2 - (J_4 J_3)J_4]\psi(x).
\end{aligned}$$

For the q-hypergeometric orthogonal polynomials $Y_n(x)$, $x \neq \cos(\theta)$, we use the notation Q where

$$Q = \begin{cases} r - s, & \text{if } Y_n(x) = {}_r\varphi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| q; x\right) \\[2ex] r - s - 1, & \text{if } Y_n(x) = {}_r\varphi_s\left(\begin{matrix} a_1, \ldots, a_{r-1}, x \\ b_1, \ldots, b_s \end{matrix} \middle| q; k\right) \end{cases}$$

Also we will define three cases of J_1 and J_3 depending on the weight function $W(x, q)$ of $Y_n(x)$

(1) If $W(x, q) = \prod_{i=1}^{k} e_q(f_i(x))x^\alpha$
choose

$$J_1 = -q^{-1}(q^{-1}x)^{-Q},$$
$$J_3 = \frac{-1}{x^Q}\frac{W(qx, q)}{W(x, q)}.$$

(2) If $W(x, q) = \prod_{i=1}^{k} e_q(f_i(x))\prod_{j=1}^{r} E_q(g_j(x))$
choose

$$J_1 = \frac{1}{x}\prod_{j=1}^{r}\frac{E_q(g_j(q^{-1}x))}{E_q(g_j(x))},$$

$$J_3 = \frac{1}{x}\prod_{i=1}^{k}\frac{e_q(f_i(qx))}{e_q(f_i(x))}.$$

(3) If $W(x, q) = \prod_{i=1}^{k} e_q(f_i(x))\prod_{j=1}^{r} E_q(g_j(x))x^\alpha$
choose

$$J_1 = \frac{1}{q}\prod_{j=1}^{r}\frac{E_q(g_j(q^{-1}x))}{E_q(g_j(x))},$$

$$J_3 = q^\alpha\prod_{i=1}^{k}\frac{e_q(f_i(qx))}{e_q(f_i(x))}.$$

4. *q*-DIFFERENCE EQUATIONS OF THE *q*-HYPERGEOMETRIC POLYNOMIALS $Y_n(x)$, $x \neq \cos(\theta)$

4.1. q-Laguerre Polynomial

The q-Laguerre polynomial is defined by

$$L_n^\alpha(x; q) = \frac{(q^{\alpha+1;}q)_n}{(q; q)_n}\,{}_1\varphi_1\left(\begin{matrix} q^{-n} \\ \\ q^{\alpha+1}\end{matrix}\ |q; -q^{n+\alpha+1}x\right)$$

and the weight function is

$$W(x; \alpha; q) = e_q(-x)x^\alpha.$$

Then $Q = 0$ and

$$W(qx; \alpha; q) = q^\alpha(1 + x)e_q(-x)x^\alpha.$$

Now J_1 and J_3 are

$$J_1 = -q^{-1},$$
$$J_3 = -q^{\alpha}(1+x).$$

Then the q-difference equation of $L_n^{\alpha}(x;q)$ is

$$(-q^{-1}D_{q^{-1}} + q^{\alpha}(x+1)D_q)L_n^{\alpha}(x;q) = \lambda_q[n]_q L_n^{\alpha}(x;q).$$

By equating the coefficients of x^n one gets

$$q^{\alpha}[n]_q = \lambda_q[n]_q,$$

then

$$\lambda_q = q^{\alpha}.$$

4.2. Stieltjes–Wigert Polynomial

The Stieltjes–Wigert polynomial is defined by

$$S_n(x;q) = \frac{1}{(q;q)_n}\,{}_1\varphi_1\left(\begin{matrix} q^{-n} & \\ & |q; -q^{n+1}x \\ 0 & \end{matrix}\right)$$

and the weight function is

$$W(x;q) = e_q(-x)e_q(-qx^{-1}).$$

Then $Q = 0$ and

$$W(qx;q) = xe_q(-x)e_q(-qx^{-1}).$$

Now J_1 and J_3 are

$$J_1 = -q^{-1},$$
$$J_3 = -x.$$

Then the q-difference equation of $S_n(x;q)$ is

$$(-q^{-1}D_{q^{-1}} + xD_q)S_n(x;q) = \lambda_q[n]_q S_n(x;q).$$

By equating the coefficients of x^n one gets

$$[n]_q = \lambda_q[n]_q,$$

then

$$\lambda_q = 1.$$

4.3. *q*-Hermite II Polynomial

The q-Hermite II polynomial is defined by

$$h_n(x;q) = i^{-n} q^{-\binom{n}{2}} {}_2\varphi_0 \left(\begin{matrix} q^{-n}, ix & \\ & |q; -q^n \\ - & \end{matrix} \right)$$

and the weight function is

$$W(x;q) = e_q(ix)e_q(-ix).$$

Then $Q = 1$ and

$$W(qx;q) = (1+x^2)e_q(ix)e_q(-ix).$$

Now J_1 and J_3 are

$$J_1 = -x^{-1},$$
$$J_3 = -(1+x^2)x^{-1}.$$

Then the q-difference equation is

$$(-x^{-1}D_{q^{-1}} + (1+x^2)x^{-1}D_q)h_n(x;q) = \lambda_q [n]_q h_n(x;q).$$

By equating the coefficients of x^n one gets

$$[n]_q = \lambda_q [n]_q,$$

then

$$\lambda_q = 1.$$

4.4. Al-Salam-Carlitz II

The Al-Salam-Carlitz II is defined by

$$V_n^a(x;q) = (-a)^n q^{-\binom{n}{2}} {}_2\varphi_0 \left(\begin{matrix} q^{-n}, x & \\ & |q; \frac{q^n}{a} \\ - & \end{matrix} \right)$$

and the weight function is

$$W(x;a;q) = e_q(x)e_q(a^{-1}x).$$

Then $Q = 1$ and

$$W(qx;a;q) = \frac{(1-x)(a-x)}{a} e_q(x)e_q(a^{-1}x).$$

Now J_1 and J_3 are

$$J_1 = -x^{-1},$$
$$J_3 = -\frac{(1-x)(a-x)}{a}x^{-1}.$$

Then the q-difference equation of $V_n^a(x;q)$ is

$$\left(-x^{-1}D_{q^{-1}} + \frac{(1-x)(a-x)}{a}x^{-1}D_q\right)V_n^a(x;q) = \lambda_q[n]_q V_n^a(x;q).$$

By equating the coefficients of x^n one gets

$$\frac{1}{a}[n]_q = \lambda_q[n]_q,$$

then

$$\lambda_q = \frac{1}{a}.$$

4.5. Al-Salam-Carlitz I

The Al-Salam-Carlitz I is defined by

$$U_n^a(x;q) = (-a)^n q^{\binom{n}{2}} {}_2\varphi_1\left(\begin{matrix} q^{-n}, x^{-1} & \\ & |q; \frac{qx}{a} \\ 0 & \end{matrix}\right),$$

and the weight function is

$$W(x;a;q) = E_q(-qx)E_q(-a^{-1}qx).$$

Then

$$\frac{E_q(-q^{-1}qx)E_q(-q^{-1}a^{-1}qx)}{E_q(-qx)E_q(-a^{-1}qx)} = \frac{(1-x)(a-x)}{a}.$$

Now J_1 and J_3 are

$$J_1 = \frac{(1-x)(a-x)}{ax},$$
$$J_3 = \frac{1}{x}.$$

Then the q-difference equation of $U_n^a(x;q)$ is

$$\left(\frac{(1-x)(a-x)}{ax}D_{q^{-1}} - \frac{1}{x}D_q\right)U_n^a(x;q) = \lambda_q[n]_q U_n^a(x;q).$$

By equating the coefficients of x^n one gets

$$\frac{1}{a}[n]_{q^{-1}} = \lambda_q [n]_q,$$

then

$$\lambda_q = \frac{1}{a} q^{1-n}.$$

4.6. Big q-Jacobi

The Big q-Jacobi is defined by

$$P_n(x; a, b, c; q) = {}_3\varphi_2 \begin{pmatrix} q^{-n}, abq^{n+1}, x & \\ & |q; q \\ aq, cq & \end{pmatrix}$$

and the weight function is

$$W(x; a, b, c; q) = e_q(x) e_q(-bc^{-1}x) E_q(-a^{-1}x) E_q(-c^{-1}x).$$

Then

$$\frac{E_q(-q^{-1}a^{-1}x)E_q(-q^{-1}c^{-1}x)}{E_q(-a^{-1}x)E_q(-c^{-1}x)} = \frac{(qa-x)(qc-x)}{q^2ac},$$

$$\frac{e_q(qx)e_q(-qbc^{-1}x)}{e_q(x)e_q(-bc^{-1}x)} = \frac{(1-x)(c-bx)}{c}.$$

Now J_1 and J_3 are

$$J_1 = \frac{(qa-x)(qc-x)}{q^2acx},$$

$$J_3 = \frac{(1-x)(c-bx)}{cx}.$$

Then the q-difference equation of $P_n(x; a, b, c; q)$ is

$$\left(\frac{(qa-x)(qc-x)}{q^2acx} D_{q^{-1}} - \frac{(1-x)(c-bx)}{cx} D_q \right) \times P_n(x; a, b, c; q) = \lambda_q [n]_q P_n(x; a, b, c; q).$$

By equating the coefficients of x^n one gets

$$\frac{1}{q^2ac}[n]_{q^{-1}} - \frac{b}{c}[n]_q = \lambda_q [n]_q,$$

then

$$\lambda_q = \frac{1 - q^{n+1}ab}{q^{n+1}ac}.$$

4.7. Big *q*-Laguerre

The Big *q*-Lagurre is defined by

$$P_n(x; a, b; q) = {}_3\varphi_2 \left(\begin{matrix} q^{-n}, 0, x \\ \\ aq, bq \end{matrix} \;|q; q \right)$$

and the weight function is

$$W(x; a, b; q) = e_q(x) E_q(-a^{-1}x) E_q(-b^{-1}x).$$

Then

$$\frac{E_q(-q^{-1}a^{-1}x)E_q(-q^{-1}b^{-1}x)}{E_q(-a^{-1}x)E_q(-b^{-1}x)} = \frac{(qb - x)(qa - x)}{q^2ab},$$

$$\frac{e_q(qx)}{e_q(x)} = (1 - x).$$

Now J_1 and J_3 are

$$J_1 = \frac{(qb - x)(qa - x)}{q^2abx},$$

$$J_3 = \frac{(1 - x)}{x}.$$

Then the *q*-difference equation of $P_n(x; a, b; q)$ is

$$\left(\frac{(qb - x)(qa - x)}{q^2abx} D_{q^{-1}} - \frac{(1 - x)}{x} D_q \right) P_n(x; a, b; q) = \lambda_q [n]_q P_n(x; a, b; q).$$

By equating the coefficients of x^n one gets

$$\frac{1}{q^2ab}[n]_{q^{-1}} = \lambda_q [n]_q,$$

then

$$\lambda_q = \frac{1}{q^{n+1}ab}.$$

4.8. Discrete q-Hermite I Polynomial

The discrete q-Hermite I polynomial is defined by

$$h_n(x;q) = q^{\binom{n}{2}} {}_2\varphi_1\left(\begin{matrix} q^{-n}, x^{-1} & \\ & |q; -qx \\ 0 & \end{matrix}\right)$$

and the weight function is

$$W(x;q) = E_q(-qx)E_q(qx).$$

Then

$$\frac{E_q(-q^{-1}qx)E_q(q^{-1}qx)}{E_q(-qx)E_q(qx)} = (1-x^2).$$

Now J_1 and J_3 are

$$J_1 = \frac{(1-x^2)}{x},$$
$$J_3 = \frac{1}{x}.$$

Then the q-difference equation of $h_n(x;q)$ is

$$\left(\frac{(1-x^2)}{x}D_{q^{-1}} - \frac{1}{x}D_q\right)h_n(x;q) = \lambda_q[n]_q h_n(x;q).$$

By equating the coefficients of x^n one gets

$$-[n]_{q^{-1}} = \lambda_q[n]_q,$$

then

$$\lambda_q = -q^{1-n}.$$

4.9. Little q-Jacobi Polynomial

The Little q-Jacobi polynomial is defined by

$$P_n(x;a,b|q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^{n+1} & \\ & |q; qx \\ aq & \end{matrix}\right)$$

and the weight function is

$$W(x;\alpha,\beta|q;q) = e_q(q^{\beta+1}x)E_q(-qx)x^{\alpha}.$$

Then

$$\frac{E_q(-q^{-1}qx)}{E_q(-qx)} = (1-x),$$

$$\frac{e_q(qq^{\beta+1}x)}{e_q(q^{\beta+1}x)} = (1-q^{\beta+1}x).$$

Now J_1 and J_3 are

$$J_1 = \frac{1-x}{q},$$

$$J_3 = (1-q^{\beta+1}x)q^{\alpha}.$$

Then the q-difference equation of $P_n(x;a,b|q)$ is

$$\left(\frac{(1-x)}{q}D_{q^{-1}} - (1-q^{\beta+1}x)q^{\alpha}D_q\right)P_n(x;a,b|q) = \lambda_q[n]_q P_n(x;a,b|q).$$

By equating the coefficients of x^n one gets

$$-\frac{1}{q}[n]_{q^{-1}} + q^{\alpha+\beta+1}[n]_q = \lambda_q[n]_q,$$

then

$$\lambda_q = q^{\alpha+\beta+1} - q^{-n}.$$

4.10. Little *q*-Laguerre/Wall Polynomial

The Little q-Lagurre/Wall polynomial is defined by

$$P_n(x;a|q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, 0 & \\ & |q;qx \\ aq & \end{matrix}\right)$$

and the weight function is

$$W(x;\alpha|q;q) = E_q(-qx)x^{\alpha}.$$

Then

$$\frac{E_q(-q^{-1}qx)}{E_q(-qx)} = (1-x).$$

Now J_1 and J_3 are

$$J_1 = \frac{(1-x)}{q},$$

$$J_3 = q^{\alpha}.$$

Then the q-difference equation of $P_n(x; a|q)$ is

$$\left(\frac{(1-x)}{q} D_{q^{-1}} - q^{\alpha} D_q\right) P_n(x;\, a|q) = \lambda_q [n]_q P_n(x;\, a|q).$$

By equating the coefficients of x^n one gets

$$-\frac{1}{q}[n]_{q^{-1}} = \lambda_q [n]_q,$$

then

$$\lambda_q = -q^{-n}.$$

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